A UNIFIED MATHEMATICAL MODEL OF THICKENING, FILTRATION AND CENTRIFUGATION OF FLOCULATED SUSPENSIONS

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ABSTRACT

We present a phenomenological theory of solid-liquid separation processes, including thickening, centrifugation and filtration, of floculated suspensions. This theory starts from the mass and linear momentum balances for the solid and liquid components. Introducing constitutive assumptions, describing the material by two phenomenological functions and performing a dimensional analysis leads to a system of three equations for the mixture flow velocity, the pore pressure, and the solid concentration.

In one space dimension, the model gives rise to initial-boundary value problems of hyperbolic-parabolic degenerate partial differential equations. A brief account of existence and uniqueness results and suitable numerical schemes is presented. Predictions of the phenomenological theory are illustrated by numerical examples.

INTRODUCTION

Thickening, filtration and centrifugation are important stages in the beneficiation of ores, in the production of chemicals, pulp and paper, food and many other industries. All of them are specific processes of the general field of solid-liquid separation and particle technology and therefore have many features in common. The basic principle underlying these processes is the relative flow of particles and fluids. In some cases isolated particles move through the fluid and in others the fluid moves along the particles forming a more or less consolidated network. Despite their common features, each of these processes have been developed independently by different groups of researchers with different technological interest. The developing of sedimentation and thickening comes, since the pioneering work of Coe and Clevenger (1916) to our recent work (Bustos et. al. 1999), from the mining industry, while filtration and centrifugation was developed in the food and chemical industry.

In this paper we present a unified theory of solid-liquid separation processes, including sedimentation, filtration and centrifugation. We show that these processes are described by the volume fraction and the pressure of each component, solid and fluid, and by the solid-fluid interaction force. These variables obey a hyperbolic-parabolic strongly degenerate partial differential equation with initial and boundary conditions. A particular process can then be simulated by solving this equation numerically.

The benefits of this formulation are then illustrated by a brief account of some very recent existence and uniqueness results (Bürger et al. 2000c) related to the equation for the thickening problem. The mathematical analysis confirms the well-posedness of the mathematical model and is essential for the design of appropriate numerical solution procedures (Bürger et al. 2000d). A limited choice of calculated examples from thickening, centrifugation and filtration illustrates the theory.

CONSTITUTIVE ASSUMPTIONS AND MODEL EQUATIONS

The phenomenological theory of solid-liquid separation processes describes the separation of floculated suspensions under the influence of gravity, centrifugal force or applied pressure. A solid-liquid suspension is considered here as a mixture of two superimposed continuous media. The starting point of the modeling are the mass and linear momentum balances of both components. Constitutive assumptions, an order-of-magnitude study and the restriction to one space dimension reduce these equations to one scalar partial differential equation for the volumetric solids concentration plus an algebraic relationship for the excess pore pressure. The unusual mathematical property is the
hyperbolic-parabolic type of that equation, where the type-change interface is determined by the solution, i.e. its location is unknown a priori.

Although Kynch’s kinematical sedimentation theory (Kynch, 1952) describes well many phenomena, especially the gravitational or centrifugal settling of suspensions, its major shortcoming is its prediction that concentrations always propagate along straight characteristics (which might, of course, intersect and thus produce discontinuities). This prediction contradicts observations of flocculated slurries, which form compressible sediments with curved iso-concentration lines. The phenomenological theory models correctly this behaviour and at the same time produces a mathematical model that can be solved easily and therefore be implemented as a simulation tool.

We assume that the solid particles are small with respect to the containing vessel and have the same density \( \rho_s \); that the constituents of the suspension are incompressible; that the suspension is completely flocculated before the sedimentation begins; and that there is no mass transfer between the solid and the fluid. Then the mixture can be described by the local solids volume fraction \( \phi \), the solid and fluid phase velocities \( v_s \) and \( v_f \) and Cauchy stress tensors \( T_s \) and \( T_f \), the solid-fluid interaction force per unit volume \( m \), and the body forces \( b_s = b_f = -gk \) in the gravitational case, with \( k \) the upwards-pointing unit vector, or \( b_s = \omega \times \omega \times r - 2\omega \times v_s \), \( c \in \{s,f\} \) in the centrifugal, where \( \omega \) is the angular velocity and \( r \) the length of the radius vector.

With the volume-average velocity described by \( q = \phi v_s + (1-\phi)v_f \), the respective local mass balances for the solid and for the mixture are
\[
\tilde{\epsilon} \phi + \nabla \cdot (\phi v_s) = 0, \quad \nabla \cdot q = 0. \tag{1}
\]
The solid and liquid component linear momentum balances are
\[
\rho_s \phi D_s v_s = \nabla \cdot T_s + \rho_s \phi b_s + m, \tag{2}
\]
\[
\rho_f (1-\phi) D_f v_f = \nabla \cdot T_f + \rho_f (1-\phi)b_f - m. \tag{3}
\]
We assume that the Cauchy stress tensors take the form
\[
T_c = \rho_s I + T_c^E, \quad c \in \{s,f\}, \quad \text{where } \rho_s \text{ and } \rho_f \text{ are the phase pressures, and that the viscous solid and fluid stress tensors are given by } T_c^E = \mu_c \left[ \nabla v_c + (\nabla v_c)^T - \frac{2}{3} (\nabla \cdot v_c) I \right] \text{ for } c \in \{s,f\}. \]
The phase shear viscosities \( \mu_s(\phi) \) and \( \mu_f(\phi) \) or alternatively, a mixture-representative viscosity \( \mu(\phi) = \mu_s(\phi) + \mu_f(\phi) \) and the ratio \( \eta(\phi) = \mu(\phi)/\mu_f(\phi) \) are given constitutive functions. For simplicity, we assume here that bulk viscosities vanish.

The theoretical variables \( p_f \) and \( p_s \) are replaced by the pore pressure \( p \) and the effective solid stress \( \sigma_e \), which can be measured experimentally. The assumption that the surface porosity of the sediment equals its volume porosity implies the relationships \( p_f = (1 - \phi) p \) and \( p_s = p + \sigma_e \).

We assume that the solid flocs begin to touch each other at a critical concentration \( \phi_c \), then the constitutive equation for \( \sigma_e \) satisfies \( d\sigma_e(\phi)/d\phi = 0 \) for \( \phi \leq \phi_c \) and \( d\sigma_e(\phi)/d\phi > 0 \) for \( \phi > \phi_c \).

Since the flow depends only on the pore pressure in excess of the hydrostatic pressure, the excess pore pressure \( p_e \) is used instead of \( p \). That quantity is defined by \( p_e = p - p_s g (H - z) \) in the gravitational case, where \( H \) is the height of the settling vessel, and by \( p_e = p - p_s \Sigma g \rho c \sigma^2 r^2 \) for centrifugation, where \( \sigma \) is the scalar angular velocity and \( r \) the length of the radius vector.

The solid-fluid interaction force \( m \) is decomposed into a hydrostatic part \( m_h \) and a dynamic part \( m_d \), i.e. \( m = m_h + m_d \), where \( m_h = \beta \psi \phi \) and \( m_d = \alpha(\phi) v_f + \gamma(\phi) D_f v_f \). Here \( v_f = v_s - v_f \) denotes the solid-liquid relative or drift velocity and \( \alpha(\phi) \) is the resistance coefficient, which is given as the second material constitutive function. Taking the liquid component linear momentum balance at equilibrium shows that \( \beta = p \). The virtual mass \( \gamma(\phi) \) will not be determined since we anticipate that the term \( D_f v_f \) can be neglected.

It is convenient to replace \( \alpha(\phi) \) by the Kynch batch flux density function \( \phi(\phi) = -\delta(1-\phi)^2/\alpha(\phi) \), where \( \delta = \rho_s - \rho_f \). Usually \( f \) is assumed to be a piecewise smooth function satisfying \( \phi(\phi) = 0 \) for \( 0 < \phi < \phi_{\text{max}} \), where \( \phi_{\text{max}} \leq 1 \) is the maximum concentration, and \( f(\phi) = 0 \) otherwise.

The final multidimensional model equations will not be solved in this paper, therefore their derivation will be limited here to the gravitational case, see Bürger and Concha (2000) for the centrifugal case. Introducing the present assumptions into the linear momentum balances yields
\[
v_f = f(\phi) \frac{\epsilon}{\delta_0(1-\phi)} \left[ \nabla \sigma_0(\phi) + \phi \delta_0 g k + \phi \nabla \cdot T_s \right],
\]
\[
-\nabla \cdot T_f + \phi(\phi) D_f v_f - m_f - \frac{\rho_f}{1-\phi} \left[ \phi v_f - (1-\phi) v_f \right] - \frac{\gamma(\phi)}{1-\phi} \nabla \cdot D_f v_f
\]
We assume that \( 10^{-4} \text{m/s}, 1 \text{m} \) and \( 10^{-6} \text{m}^2/\text{s} \) are typical values of the settling velocity of a single floc in pure liquid, of the height of the separation vessel and of the kinematic viscosity of the pure fluid, respectively. Then a dimensional analysis justifies to reduce (4) to
This equation replaces one of the component linear momentum balances. The second of these two balances appears as the linear momentum balance of the mixture,

\[
-p(<J>)D,q + \mu(<J>)t.q - Vp_e = Vq(<J>) + \rho(<J>)a_k
\]

where \(\rho(<J>) = \rho_0 <J> + \rho_s (1 - <J>)\) is the local density of the mixture, \(\rho(<J>) = \delta_1 (1 - <J>/\rho_0)\) and \(s\) is a particular known function of \(<J>\) and its partial derivatives of up to third order (Bürger et al. 2000e). The function \(\rho\) describes the interaction of the concentration field \(<J>\) with the average flow field \(q\) and the excess pore pressure \(p_e\) coming from viscous stress due to concentration inhomogeneities and from advective acceleration due to solid-liquid relative motion and diffusion stress.

With

\[
a(<J>) = \frac{f(<J>)}{\delta_0 g} \sigma_a(<J>), \quad A(<J>) = \int_0^x a(s)ds,
\]

the continuity equations read for the gravitational case

\[
\partial_t <J> + \nabla \cdot (q(<J>) + f(<J>)k) = \Delta A(<J>), \quad \nabla \cdot q = 0.
\]

From the assumptions on \(a(<J>)\) and \(f_c(<J>)\), we see that

\[
a(<J>) = \begin{cases} 
0 & \text{for } <J> \leq <J>_s, \ \text{ph}\geq <J>_{\text{max}}; \ \\
>0 & \text{for } <J>_s < <J> < <J>_{\text{max}}; \ \\
\end{cases}
\]

The mathematical and numerical difficulties due to this type degeneracy are even exacerbated by the frequent assumption that the derivative of \(\sigma_a\) and hence the diffusion function \(a(<J>)\) in our theory, is discontinuous at \(<J> = <J>_s\). After inserting (4) into (5), the final set of field equations for the determination of \(<J>, q\) and \(p_e\) is given by (5) and (6).

**ONE-DIMENSIONAL SOLID-LIQUID SEPARATION MODELS**

In the spatially one-dimensional case, we obtain \(q = q(t)\). Using Eq. (5) the excess pore pressure can be calculated a posteriori from the concentration distribution. We may therefore neglect both viscous and advective acceleration terms to obtain the governing equations

\[
\partial_t <J> + \partial_z (q(<J>) + f(<J>)) = a_z^2 A(<J>),
\]

\[
\partial_z p_e = -\partial_z \sigma_a(<J>) - \partial_z g <J>.
\]

Figure 1: a) Rotating tube (\(\gamma = 0\)), b) axisymmetric centrifuge (\(\gamma = 1\)).

In the centrifugal case, with \(r\) being the radius and neglecting Coriolis effects, Eqs. (7) and (8) are replaced by

\[
\partial_t <J> + \partial_z \left[\frac{f(<J>)\omega^2}{g} - \gamma A(<J>/r)\right]
\]

\[
= \partial_z A(<J>) + \gamma \left[\frac{f(<J>)\omega^2}{g} + A(<J>/r)\right],
\]

\[
\partial_z p_e = \delta_0 \omega^2 r - \partial_z \sigma_a(<J>).
\]

The parameter \(\gamma\) takes the values indicated in Figure 1.

Observe that it is sufficient to solve the scalar equation Eq. (7) or (9), and to use Eq. (8) and (10), respectively, to calculate the local excess pore pressure. We now specify initial and boundary conditions for Eqs. (7) and (9).

**Gravity sedimentation**

Consider first batch settling and continuous thickening in an ideal thickener of height \(L\). The velocity \((1)\) \(= 0\) is given by discharge control. The case \(q = 0\) corresponds to batch settling in a closed column. We assume that an initial concentration distribution

\[
<J>(z,0) = \phi_0(z), \quad 0 \leq z \leq L
\]

in the column is given. The boundary condition at \(z = 0\),

\[
(f(<J>) - \partial_z A(<J>))(0,t) = 0, \quad t > 0,
\]
comes from reducing at \( z = 0 \) the total solids volume flux \( \partial \tau \phi + \partial \phi - \Delta \phi \) to the convective part \( \partial \tau \phi \). At \( z = L \), a solids feed flux \( \psi \) is prescribed,
\[
(q(t) + f(\phi) - \partial_A(\phi))(L,t) = \psi(t), \quad t > 0, \tag{13}
\]
where \( \psi = 0 \) corresponds to the batch case.

**Batch centrifugation**

We assume that the radius \( r \) varies between an inner radius \( R_0 \) and an outer radius \( R \). We then obtain the initial and boundary conditions
\[
\phi(r, 0) = \phi_0(r), \quad R_0 \leq r \leq R, \tag{14}
\]
\[
(\alpha f(\phi) + g - \partial_A(\phi))(r_0, t) = 0, \quad r_0 \in [R_0, R], \quad t > 0. \tag{15}
\]

Figure 2: Pressure filtration of a suspension

**Pressure filtration**

This process follows the stages sketched in Figure 2. The filtration column has at its bottom a filter medium which lets only the liquid pass. Its top \( h = h(t) \) is represented by a piston which can move downwards due to an applied pressure \( \sigma(t) \). Both, the filter medium and the filter cake, formed by settling of the initially suspended solids, exert resistance to the flow of the filtrate and thereby to the movement of the piston. Obviously, the volume average mixture velocity is now \( q(t) = h'(t) \), whose absolute value at the same time is the filtrate volume flow rate, and equation (7) is considered for \( t > 0 \) on the time-dependent height interval \( 0 < z < h(t) \). Since \( h(t) \) has to be determined simultaneously with the sought solution \( \phi \), we arrive at a free boundary problem. The solids phase velocity in heights zero and \( h(t) \) should take the respective values zero and \( h'(t) \), which implies the kinematic boundary conditions
\[
(f(\phi) - \partial_A(\phi))(0, t) = h'(t) \phi(0, t), \tag{16}
\]
\[
(f(\phi) - \partial_A(\phi))(h(t), t) = 0, \quad t > 0. \tag{17}
\]

An additional condition is necessary to describe the coupling between the applied pressure \( \sigma(t) \) and the filtrate rate \( h'(t) \). A force balance yields the equation
\[
\sigma(t) = \sigma(\phi(0, t)) - [m + \rho_r(h(t) - R_0)] - \mu_f R_m h'(t). \tag{18}
\]

The governing equations, Eqns. (7) and (9), are highly non-standard partial differential equations. In particular, because of their nonlinear nature and mixed hyperbolic-parabolic type, their solutions may become discontinuous even if the initial concentration is smooth. Consequently, the solutions have to be interpreted in a weak sense. However, weak solutions are not uniquely determined by their initial and boundary data, and additional selection criteria, or so-called entropy conditions, are needed to single out the unique physically relevant solution. We refer to Bürger and Karlsen (2001), Bürger et al. (2000) and to Espedal and Karlsen (2000) for details on the mathematical background of strongly degenerate nonlinear partial differential equations.

The nonlinear nature of mathematical models derived herein rules out analytical solution techniques and one has to resort to numerical methods. When designing numerical methods for the one-dimensional solid-liquid separation models one should be a bit careful. It was observed in Evje and Karlsen (2000) that numerical methods, based on “naive” finite difference discretization may produce stable but physically wrong solutions. This feature is intimately connected to the fact that one has to impose so-called entropy conditions in the continuous model to single out its unique physically relevant solution. Consequently, when designing numerical methods one has to make sure that the proposed methods have these entropy conditions or at least discrete versions of them “built in”.

As it is outside the scope of this paper to explain in detail why the numerical method employed for the calculation of the examples presented here, i.e. the upwind method as studied by Bürger and Karlsen (2001), produces correct solutions, let us simply say that this is (roughly speaking) related to the conservative form and the upwind nature of the method. In particular, the upwind nature implies that the method obeys a sort of discrete entropy condition. We refer to the papers cited in this section and to Karlsen and Risebro (2000) for details.
NUMERICAL EXAMPLES

The numerical examples presented here have been adopted from several papers published previously by the authors and are solely meant to illustrate the predictions of the unified mathematical theory. For new calculations comparing the predictions of the model with experimental results from centrifugation and filtration we refer to our companion contribution (Garrido et al. 2001). The model predictions for batch sedimentation have extensively been correlated with experimental data in the papers by Bürger et al. (2000b) and Garrido et al. (2000).

We first consider a batch settling experiment of a kaolin suspension in a column reported by Tiller et al. (1991). For that material, the model functions

\[ f(\phi) = -2.7 \times 10^{-4} (1 - 2\phi)^{1.5} \text{ m/s,} \]

\[ \sigma_c(\phi) = \begin{cases} 0 & \text{for } \phi \leq \phi_c = 0.07, \\ 1.2 \left( \frac{\phi}{\phi_c} \right)^5 - 1 & \text{for } \phi > \phi_c, \end{cases} \text{ Pa} \]

were found suitable, see Bürger et al. (2000b). Fig. 3 shows a simulation of the settling experiment from Tiller et al. (1991), together with the published measured iso-concentration lines, obtained by solving the initial-boundary value problem formed by Eqns. (7) and (11)-(13) by a second-order upwind finite difference method, see Bürger and Karlsen (2001). Figure 3 shows the curved iso-concentration lines of the sediment (where \( \phi > \phi_c \)). These lines emerge successively from the bottom, i.e., the bottom concentration is slowly increasing.

A similar method can be employed to solve Eqns. (9), (14) and (16) for the centrifugation problem. Consider the continuous constitutive functions

\[ f(\phi) = 10^{-16} \text{ m/s} \times \begin{cases} 2320\phi^2 - 217\phi & \text{for } 0 < \phi \leq 0.02, \\ -58.56\phi^{0.727} & \text{for } 0.02 < \phi \leq 0.056, \\ -1.387\phi^{-0.571} & \text{for } 0.056 < \phi \leq 0.107, \\ -6.68\phi^{0.32} & \text{for } 0.107 < \phi \leq \phi_{\text{max}} = 0.5, \end{cases} \]

\[ \sigma_c(\phi) = \begin{cases} 0 & \text{for } \phi \leq \phi_c = 0.14, \\ 1.142 \times 10^5 (\phi - \phi_c)^{1.667} & \text{for } \phi > \phi_c, \end{cases} \text{ Pa} \]

determined for a limestone suspension by Bürger and Karlsen (2000) using measurements by Sambuchi et al. (1991). Figure 4 shows the numerical simulation by both a settling plot and selected concentration profiles as well as the remaining parameters.

It is well known (Anestis and Schneider 1983) that in both centrifugal cases, \( \gamma = 0 \) and \( \gamma = 1 \), and unlike the gravity case, characteristics and iso-concentration lines in the hindered settling zone (where \( \phi \leq \phi_c \)) do not coincide. As a consequence, as can be seen from the vertical iso-concentration lines, the concentration of the bulk suspension between the slurry/supernate and the slurry/sediment interfaces decreases with time, and the slurry/supernate interface has curved shape.
Figure 5: Simulation of pressure filtration of a kaolin suspension: filtration plot (top) and concentration profiles (bottom). The fat dots in the second diagram denote the concentration at height $h(t)$ and are plotted in time intervals of length $\Delta r$.

Another variant of that numerical method has been used to calculate a pressure filtration process of a flocculated kaolin suspension. For that material, the functions

\[
 f(\phi) = \begin{cases} 
 -1.41 \times 10^{-4} \phi(1 - \phi)^{28.88} \text{ m/s for } \phi \leq 0.14, \\
 -2.25 \times 10^{-11} \phi^{-1.1} \text{ m/s for } \phi \geq 0.32, \\
 \mathcal{I}(\phi) \text{ otherwise} 
\end{cases}
\]

\[
 \sigma_s(\phi) = \begin{cases} 
 0 \text{ for } \phi \leq \phi_c = 0.32, \\
 19000 \left( \frac{\phi}{\phi_c} \right)^{1.1} - 1 \text{ Pa for } \phi > \phi_c 
\end{cases}
\]

were determined, where $\mathcal{I}(\phi)$ is a particular smooth interpolant and the segments of both functions for $\phi > \phi_c$ have been converted from well-known formulae relating permeability, solid stress and porosity advanced by Tiller and Leu (1980). Figure 5 shows a filtration plot and a sequence of concentration profiles for a simulated filtration process, followed by expression of the filter cake. The remaining parameters were $\delta_p = 1618.2 \text{ kg/m}^3$, $\mu = 9.78 \times 10^4 \text{ Pa s}$ and $P_f = 3 \times 10^{10} \text{ m}^{-1}$. 

Observe in Figure 5 that a zone of clear liquid is forming beneath the moving piston, and that the filtrate rate decreases during the growth of the filter cake. Very soon after the slurry-supernate interface has reached the cake level, the composition of the cake remains constant and so does the filtrate rate due to Eq. (16). After 4300 seconds, the piston touches the filter cake, which is then further compressed until its solids concentration is uniform.

**CONCLUSIONS**

We outlined in this paper a unified theory of solid-liquid separation processes of flocculated suspensions. Similar unifying theories have been proposed previously (Tiller and Hsyung 1993, Landman and White 1994). The novelty of the present work, and the authors’ papers cited herein which are outside the scope of this contribution, consists in the simulation of work by many other researchers, whose results are satisfactorily reproduced. A selection of detailed comparisons with experimental findings will be presented in our companion paper (Garrido et al. 2001).

At the same time, new results of mathematical analysis of the model equations are available. The results of this research confirm the soundness of the phenomenological framework and have made the development of appropriate numerical methods possible. In particular, it has turned out that sediment/suspension interfaces, such as the filter cake surface, are approximated correctly when the unique degenerating second-order field equation is solved appropriately, i.e. these interfaces do not have to be tracked explicitly. This contrasts with the treatment proposed by Tiller and Hsyung (1993), who claim that it is necessary to use different field equations and a moving boundary.

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REFERENCES


